

Additional Constants of Motion for a Discretization of the Calogero–Moser Model

Hideaki UJINO^{1*}, Luc VINET^{2†}, Tetsu YAJIMA^{3‡} and Haruo YOSHIDA^{4§}

¹*Gunma National College of Technology, Maebashi, Gunma 371–8530*

²*Université de Montreal, Montréal, Québec, Canada H3C 3J7*

³*Department of Information Science, Faculty of Engineering,
Utsunomiya University, Utsunomiya, Tochigi 321–8585*

⁴*National Astronomical Observatory of Japan, Mitaka, Tokyo 181–8588*

The maximal super-integrability of a discretization of the Calogero–Moser model introduced by Nijhoff and Pang is presented. An explicit formula for the additional constants of motion is given.

KEYWORDS: Calogero–Moser model, super-integrability, integrable discretization

1. Introduction

When one discretizes dynamical systems, it is hardly possible to avoid modifying the original systems. Controlling such modifications is thus a central problem in numerical analysis.¹ It would be ideal if a discretization conserves whole the structure of the original dynamical system such as orbits in the phase space, constants of motion, integrability and so on. As an example of such ideal discretizations, a discretization of the Kepler problem, which keeps all the constants of motion and the orbits in the phase space, was discovered.^{2,3} The Kepler problem is an integrable system that has a set of mutually independent and Poisson commutative constants of motion, whose number is the same as the degrees of freedom of the system. A dynamical system of N -degrees of freedom which has mutually independent $2N - 1$ constants of motion in the form of single-valued functions is called maximally super-integrable and so is the Kepler problem. The above discretization conserves super-integrability of the Kepler problem.

Among the family of one-dimensional integrable systems with inverse-square interactions called the Calogero–Moser–Sutherland models,⁴ the Calogero model,⁵ which is the root of the family, the Calogero–Moser model⁶ of the rational and hyperbolic types are known to be maximally super-integrable.^{7–9} What we discuss here is the super-integrability of a discretization of the rational Calogero–Moser model, which is a classical dynamical system whose Hamiltonian

*E-mail address: ujino@nat.gunma-ct.ac.jp

†E-mail address: luc.vinet@umontreal.ca

‡E-mail address: yajimat@is.utsunomiya-u.ac.jp

§E-mail address: h.yoshida@nao.ac.jp

is given by

$$H := \frac{1}{2} \sum_{i=1}^N p_i^2 - \frac{1}{2} \sum_{\substack{i,j=1 \\ i \neq j}}^N \frac{\gamma^2}{(x_i - x_j)^2}, \quad (1.1)$$

where γ , N , $p_i := p_i(t)$ and $x_i := x_i(t)$ are the coupling parameter, the number of particles, the momentum and the coordinate of the i -th particle at the time t , respectively.

It will be no exaggeration to say that the Calogero–Moser model represents the models of Calogero–Moser–Sutherland type since the Lax formulation and a systematic construction of the constants of motion for the model was discovered earlier than those for any other models of the family.⁶ Moser constructed a set of N constants of motion that are independent of each other. Later, mutual Poisson commutativity of the constants of motion of Moser-type was proved and the integrability of the model in Liouville’s sense was thus established.^{10,11} Furthermore, it turned out that the model had $N - 1$ additional constants of motion which are independent of the Moser-type ones and independent of each other as well.⁸ This concludes the maximal super-integrability of the Calogero–Moser model.

A time-discretization of the Calogero–Moser model that conserves the Moser-type constants of motion was presented by Nijhoff and Pang,¹² which was reformulated into a more convenient form by Suris.¹³ The aim of the paper is to show that the maximal super-integrability of the Calogero–Moser model holds good even after the above time-discretization: in other words, the time-discretization of the Calogero–Moser model has $N - 1$ additional constants of motion, which are independent of the Moser-type ones and independent of each other at the same time. In §2, we shall give a brief summary of the discretization of the Calogero–Moser model and its discrete Lax form. In §3, we shall explicitly construct $N - 1$ additional constants of motion of the discrete Calogero–Moser model. Concluding remarks are summarized in §4.

2. The Discrete Calogero–Moser Model

Throughout the paper, we employ Suris’ formulation of the discrete Calogero–Moser model, which is given by the following discrete symplectic map $(x_{i,n}, p_{i,n}) \rightarrow (x_{i,n+1}, p_{i,n+1})$, $i = 1, 2, \dots, N$,

$$\begin{aligned} 1 - \Delta t c_0^{-1} p_{i,n} &= \sum_{j=1}^N \frac{c_0}{x_{j,n+1} - x_{i,n} + c_0} - \sum_{\substack{j=1 \\ j \neq i}}^N \frac{c_0}{x_{j,n} - x_{i,n}}, \\ 1 - \Delta t c_0^{-1} p_{i,n+1} &= \sum_{j=1}^N \frac{c_0}{x_{i,n+1} - x_{j,n} + c_0} - \sum_{\substack{j=1 \\ j \neq i}}^N \frac{c_0}{x_{i,n+1} - x_{j,n+1}}, \end{aligned} \quad (2.1)$$

where Δt , $x_{i,n} := x_i(n\Delta t)$ and $p_{i,n} := p_i(n\Delta t)$ denote the discrete time-step, the coordinate and the momentum of the i -th particle at the n -th discrete time $n\Delta t$.¹³ The constant c_0 is

defined by $c_0^2 := -\gamma\Delta t$. In terms of the Lax pair, which consists of two $N \times N$ matrices below,

$$(L_n)_{ij} = p_{i,n}\delta_{ij} + \frac{\gamma}{x_{i,n} - x_{j,n}}(1 - \delta_{ij}), \quad (\mathcal{M}_n)_{ij} = \frac{c_0}{x_{i,n+1} - x_{j,n} + c_0}, \quad (2.2)$$

the discrete symplectic map (2.1) is expressed by the discrete Lax equation,

$$L_{n+1}\mathcal{M}_n = \mathcal{M}_n L_n, \quad (2.3)$$

which is equivalent to

$$L_{n+1} = \mathcal{M}_n L_n \mathcal{M}_n^{-1}. \quad (2.4)$$

The companion matrix \mathcal{M}_n thus plays a role of the time-evolution operator of the Lax matrix L_n . With the aid of the trace identity $\text{Tr}AB = \text{Tr}BA$ where A and B are arbitrary $N \times N$ matrices as well as the discrete Lax equation (2.3), one confirms that the trace of the power of the Lax matrix L_n satisfies

$$\text{Tr}(L_{n+1})^m = \text{Tr}(\mathcal{M}_n L_n \mathcal{M}_n^{-1})^m = \text{Tr}(L_n)^m.$$

Thus the discrete Calogero–Moser model (2.1) as well conserves the Moser-type quantities, which are exactly the same as the N constants of motion of Moser-type in the continuous time case,⁶

$$I_n^{(m)} := \text{Tr}(L_n)^m, \quad m = 1, 2, \dots, N. \quad (2.5)$$

The Moser-type quantities (2.5) are single-valued for they are rational functions of $p_{i,n}$'s and $x_{i,n}$'s. In order to confirm the mutual independence of the Moser-type quantities, all one has to do is to check their explicit forms when $\gamma = 0$,

$$I_n^{(m)} \Big|_{\gamma=0} = \sum_{i=1}^N (p_{i,n})^m, \quad (2.6)$$

which is nothing but the power sums of $p_{i,n}$'s that are indeed independent of each other. Note that the Hamiltonian (1.1) corresponds to the second constant of motion of Moser-type, $H \Big|_{t=n\Delta t} = I_n^{(2)}/2$.

The companion matrix \mathcal{M}_n of the Lax pair (2.2) satisfies another Lax equation,

$$D_{n+1}\mathcal{M}_n = \mathcal{M}_n D_n + \mathcal{M}_n \Delta t L_n (I - \Delta t c_0^{-1} L_n)^{-1}, \quad (2.7)$$

where I is the identity matrix and $D_n := \text{diag}(x_{1,n}, x_{2,n}, \dots, x_{N,n})$. The above relation (2.7) was the crucial key to the solution of the initial value problem of the discrete symplectic map (2.1). In the next section, we shall show how the relation (2.7) works in a systematic construction of $N - 1$ additional constants of motion of the discrete Calogero–Moser model (2.1).

3. Additional Constants of Motion

Our main purpose is to confirm that the $N - 1$ quantities below

$$K_n^{(m)} := \text{Tr} D_n (I - \Delta t c_0^{-1} L_n) (L_n)^{m-1} \text{Tr} L_n - \text{Tr} (L_n)^m \text{Tr} D_n (I - \Delta t c_0^{-1} L_n), \quad m = 2, 3, \dots, N, \quad (3.1)$$

are conserved by the discrete time evolution of the discrete Calogero–Moser model (2.1) and that they are independent not only of the Moser-type quantities (2.5) but also of each other. Note that the case $m = 1$ is omitted in eq. (3.1) because $K_n^{(1)} = 0$.

The discrete symplectic map (2.1) is equivalent to the discrete Lax equations (2.3) and (2.7). From the discrete Lax equations (2.3) and (2.7), one obtains

$$D_{n+1} (I - \Delta t c_0^{-1} L_{n+1}) \mathcal{M}_n = \mathcal{M}_n D_n (I - \Delta t c_0^{-1} L_n) + \mathcal{M}_n \Delta t L_n, \quad (3.2)$$

which is rewritten as

$$D_{n+1} (I - \Delta t c_0^{-1} L_{n+1}) = \mathcal{M}_n D_n (I - \Delta t c_0^{-1} L_n) \mathcal{M}_n^{-1} + \mathcal{M}_n \Delta t L_n \mathcal{M}_n^{-1}. \quad (3.3)$$

The relation (3.3) gives the time-evolution of the matrix $D_n (I - \Delta t c_0^{-1} L_n)$. Using eqs. (2.4) and (3.3) as well as the trace identity, one can perform the calculation below,

$$\begin{aligned} K_{n+1}^{(m)} &= \text{Tr} D_{n+1} (I - \Delta t c_0^{-1} L_{n+1}) (L_{n+1})^{m-1} \text{Tr} L_{n+1} \\ &\quad - \text{Tr} (L_{n+1})^m \text{Tr} D_{n+1} (I - \Delta t c_0^{-1} L_{n+1}) \\ &= \text{Tr} \mathcal{M}_n \left(D_n (I - \Delta t c_0^{-1} L_n) + \Delta t L_n \right) \mathcal{M}_n^{-1} (\mathcal{M}_n L_n \mathcal{M}_n^{-1})^{m-1} \text{Tr} \mathcal{M}_n L_n \mathcal{M}_n^{-1} \\ &\quad - \text{Tr} (\mathcal{M}_n L_n \mathcal{M}_n^{-1})^m \text{Tr} \mathcal{M}_n \left(D_n (I - \Delta t c_0^{-1} L_n) + \Delta t L_n \right) \mathcal{M}_n^{-1} \\ &= \text{Tr} D_n (I - \Delta t c_0^{-1} L_n) (L_n)^{m-1} \text{Tr} L_n - \text{Tr} (L_n)^m \text{Tr} D_n (I - \Delta t c_0^{-1} L_n) \\ &\quad + \Delta t \left(\text{Tr} (L_n)^m \text{Tr} L_n - \text{Tr} (L_n)^m \text{Tr} L_n \right) \\ &= K_n^{(m)}, \end{aligned} \quad (3.4)$$

which proves the conservation of $K_n^{(m)}$. As one can observe in the third line of eq. (3.4), cancellation of the unwanted terms derived from the second term in the r.h.s. of eq. (3.3) is crucial. The additional constants of motion (3.1), which we call the Wojciechowski-type quantities, are rational functions of $p_{i,n}$'s and $x_{i,n}$'s.

When the coupling parameter γ and the time-step Δt are zero, the $N - 1$ constants of motion $\{K_n^{(m)}\}$ (3.1) reduces to symmetric polynomials of $p_{i,n}$'s and $x_{i,n}$'s,

$$\lim_{\gamma \rightarrow 0} \lim_{\Delta t \rightarrow 0} K_n^{(m)} = \sum_{i=1}^N x_{i,n} (p_{i,n})^{m-1} \sum_{j=1}^N p_{j,n} - \sum_{i=1}^N (p_{i,n})^m \sum_{j=1}^N x_{j,n}. \quad (3.5)$$

Though it is less trivial than the mutual independence of $I_n^{(m)} \Big|_{\gamma=0}$ (2.6), the quantities (3.5) are independent of those in eq. (2.6) and independent of each other, too. Its verification is essentially the same as that for the additional constants of motion of Wojciechowski-type in

the continuous time case.⁸ Thus we find that the discrete symplectic map (2.1) has $2N - 1$ constants of motion $\{I_n^{(m)}, K_n^{(m)}\}$, which are independent of each other and single-valued as well. This concludes that the discrete symplectic map (2.1) gives not only an integrable, but a maximally super-integrable discretization of the Calogero–Moser model (1.1). This property of the discrete symplectic map (2.1) corresponds to the maximal super-integrability of the Calogero–Moser model in the continuous time case.^{6,8}

4. Concluding Remarks

The main result of the paper is the construction of the $N - 1$ additional constants of motion (3.1) besides the known N constants of motion (2.5) of the discrete symplectic map (2.1). The result concludes the maximal super-integrability of the discrete Calogero–Moser model (2.1).

It should be remarked that the $N - 1$ additional constants of motion $\{K_n^{(m)}\}$ are not exactly the same as those for the Calogero–Moser model in the continuous time case,⁸ because of the additional term proportional to Δt in their construction (3.1). In the continuous time limit $\Delta t \rightarrow 0$, however, the additional constants of motion (3.1) reduces to exactly the same additional constants of motion for the non-discrete Calogero–Moser model discovered by Wojciechowski.⁸ In other words, $K_n^{(m)}$ is a one-parameter deformation of the additional constants of motion of Wojciechowski-type in the continuous time theory. Since the orbit in the $2N$ -dimensional phase space of the maximally super-integrable model of N degrees of freedom is uniquely determined by its $2N - 1$ constants of motion, the orbit of the discrete symplectic map (2.1) in the $2N$ -dimensional phase space differs from that of the Calogero–Moser model in the continuous time case, even though both evolve from the same initial values. The former gives a one-parameter deformation of the latter.

When one deals with the Calogero–Moser model, its pairwise interactions are usually repulsive. The discrete symplectic map (2.1) with a pure imaginary γ conserves the Calogero–Moser Hamiltonian (1.1) with repulsive interactions. In this case, however, its solution becomes complex in general. Thus in the physical sense, the discrete symplectic map cannot describe a discrete version of the Calogero–Moser model with repulsive interactions. On the other hand, another super-integrable discretization of the Calogero–Moser model is given from the super-integrable discretization of the Calogero–Moser model with an external harmonic confinement.¹⁴ This discretization conserves exactly the same constants of motion of the Calogero–Moser model in the continuous time case and hence reproduces exactly the same orbit in the phase space. Repulsive interactions can be dealt with as well. Details on the comparison of the two different discretizations will be presented in a separate paper.

Acknowledgements

Most of the work was carried out during the short stay of H.U. at CRM, Université de Montréal hosted by L.V. H.U. is grateful to the warm hospitality of the institute. This author is also supported by the Grant-in-Aid for Young Scientists (B) (No. 17740259) from the Ministry of Education, Culture, Sports, Science and Technology of Japan. The work of H.Y. is partially supported by a Grant-in-Aid for Scientific Research of JSPS, No. 18540226.

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